$\boxed{11}$ Whittaker Category and Fundamental Local Equivalence

There is an equivalence of factorization cat.s

FLE $\mathrm{Whit}(G)_c \simeq KL(\check{G})_{-\check{c}}$

where c is level of G

Review G-action on Categories

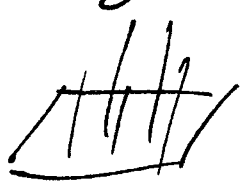
	classical (Vect)	categorical (DGCat)
algebra for S an affine derived scheme	$A \in \mathrm{Vect}$, ~~(ex. $A = \mathcal{O}(S)$)~~ $A \otimes A \to A$ $1 \in A$ ex. $A = \mathcal{O}(S)$	$\mathcal{C} \in \mathrm{DGCat}$ $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ $1_{\mathcal{C}} \in \mathcal{C}$ ex. $\mathcal{C} = QC(S) = A\text{-mod}$
module	$M \in \mathrm{Vect}$ $A \otimes M \to M$ $\Leftrightarrow$ $A \to \mathrm{End}(M)$	$M \in \mathrm{DGCat}$ $\mathcal{C} \otimes M \to M$ $\Leftrightarrow$ $\mathcal{C} \to \mathrm{End}(M)$

Ex $M \in QC(S)\text{-mod} =: \mathrm{ShvCat}/_S$

$\qquad\qquad \| $

$\quad (A\text{-mod})\text{-mod}$ $\qquad$ e.g. $U\mathfrak{g}\text{-mod}$

$A\text{-mod} \to \mathrm{End}(M)$

 $\mathrm{spec}(Z(\mathfrak{g}))$

Taking $\mathrm{End}(1)$

$$A^{op} \longrightarrow \mathrm{End}(1_{\mathrm{End}(M)})$$
$$\quad \cong \qquad\qquad \|$$
$$A \longrightarrow HC(M)$$

	Classical	Categorical
prestack $\mathcal{Y}$	$QC(\mathcal{Y}) := \lim_{S\to\mathcal{Y}} QC(S)$	$ShvCat_{/\mathcal{Y}} := \lim_{S\to\mathcal{Y}} ShvCat_{/S}$
global section	$\Gamma: QC(\mathcal{Y}) \to \Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})\text{-mod}$ $\mathcal{F} \mapsto \lim_{S\xrightarrow{f}\mathcal{Y}} \Gamma(S, f^*\mathcal{F})$ $\mathcal{O}_{\mathcal{Y}} \mapsto \lim_{S\to\mathcal{Y}} \Gamma(S,\mathcal{O}_S)$ $\overset{\shortparallel}{\{\mathcal{O}_S\}}_{S\to\mathcal{Y}} =: \Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$	$\Gamma: ShvCat_{\mathcal{Y}} \to QC(\mathcal{Y})\text{-mod}$ $\mathcal{C} \mapsto \lim_{S\to\mathcal{Y}} \Gamma(S, f^*\mathcal{C})$ $QC_{/\mathcal{Y}} \mapsto \lim \Gamma(S, f^*QC_{/\mathcal{Y}})$ $= \lim \Gamma(S, QC_{/S})$ $= QC(S)$
representation	$\mathbb{G}\text{-rep} = QC(\mathbb{B}\mathbb{G})$	$\mathbb{G}\text{-cat} = ShvCat_{/\mathbb{B}\mathbb{G}_{dR}}$
underlying	$\pi: pt \to \mathbb{B}\mathbb{G}$ $V = \pi^*\mathcal{F}$ is the underlying vector space	$\mathcal{V} \in \mathbb{G}\text{-cat}$ $\pi_{dR}^*\mathcal{V} = \mathbb{V}$ is the underlying category
invariant	$V^G = \Gamma(pt, p_*\mathcal{F})$ $p: \mathbb{B}\mathbb{G} \to pt$ $V^H = \Gamma(pt, p_*f^*\mathcal{F})$ $\mathbb{B}H \xrightarrow{f} \mathbb{B}\mathbb{G}$, $\mathbb{B}H \to pt$	$\mathbb{V}^G = \Gamma(pt, p_{dR,*}\mathcal{V})$ $\mathbb{V}^H = \Gamma(pt, p_{dR,*} f_{dR}^*\mathcal{V})$

Ex $\mathbb{G} \curvearrowright X$ $\qquad \exists$ unique map $X \to pt$.

$$(X/\mathbb{G})_{dR} \xrightarrow{f} (\mathbb{B}\mathbb{G})_{dR}$$

$$\mathcal{V} = f_* QC_{(X/\mathbb{G})_{dR}} \in ShvCat_{/\mathbb{B}\mathbb{G}_{dR}}$$

$$\begin{tikzcd} X_{dR} \arrow[r] \arrow[d, "f'"] & pt \arrow[d, "\pi"] \\ X/G \arrow[r] & \mathbb{B}G \end{tikzcd}$$

$$\begin{aligned} V &= \Gamma(pt, \pi^* V) \\ &= \Gamma(pt, \pi^* f_* QC_{(X/G)_{dR}}) \\ &= \Gamma(pt, f'_* (\pi')^* QC_{(X/G)_{dR}}) \\ &= \Gamma(X_{dR}, (\pi')^* QC_{(X/G)_{dR}}) \\ &= \Gamma(X_{dR}, QC_{X_{dR}}) \\ &= D(X) \end{aligned}$$

Ex. continued

$$\mathbb{V}^G = \Gamma(pt, p_{dR,*} f_* QC_{(X/G)_{dR}})$$

$$[(X/G)_{dR} \to \mathbb{B}G_{dR} \xrightarrow{p_{dR}} pt]$$

$$\begin{aligned} &= \Gamma((X/G)_{dR}, QC_{(X/G)_{dR}}) \\ &= D(X/G) \end{aligned}$$

In other words, $D(X)^G = D(X/G)$

$$G\text{-cat} \xrightarrow{\Gamma} D(\mathbb{B}G)\text{-mod} \quad (DG\,Cat)$$

$$\text{\rotatebox{90}{=}}$$

$$ShvCat/_{\mathbb{B}G_{dR}}$$

would be an equivalence if $\mathbb{B}G_{dR}$ is 1-affine

we don't want this, and indeed:

$\mathbb{B}G_{dR}$ is not 1-affine (i.e. $ShvCat_Y \xrightarrow{\Gamma} QC(Y)\text{-mod}\ (DG\,Cat)$ is not an equivalence

$QC(Y) \xrightarrow{\Gamma} \Gamma(Y, \mathcal{O}_Y)\text{-mod}$ is an equiv.

$\Leftrightarrow$ Y is affine

$\mathrm{ShvCat}_{/Y} \to QC(Y)\text{-mod}$ is an equiv.

$\Leftrightarrow$ Y is 1-affine

$\mathbb{B}G_{dR}$ is not 1-affine but $\mathbb{B}G$ is

If you set up cat. rep theory using $\mathbb{B}G$, not $\mathbb{B}G_{dR}$, then it is not interesting.

$$V^G \underset{\mathrm{Av}^G_*}{\overset{\mathrm{oblv}}{\rightleftarrows}} V \qquad \underline{\text{Ex}} \quad V = D(X) \qquad D(X/G) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} D(X)$$

$$X \xrightarrow{\pi} X/G$$

oblivion and averaging functors

<u>Rmk</u> G-equivariancy is a datum, not a property. $\mathcal{F} \in D(X)^G$

$$G \times X \xrightarrow{act} X \quad \Leftrightarrow \quad \mathcal{F} \in D(X) \text{ w/ } \pi^*\mathcal{F} \simeq act^*\mathcal{F}$$

$$\pi \downarrow$$

$$X$$

In usual rep. theory, one has $V^G \subseteq V$

In particular, it makes sense to ask if $v \in V$ belongs to V^G.

Not any more in our setting.

On the other hand, if G is contractible (e.g. $N \in B$ unipotent group), then G-equivariancy is a property.

Khazhdan-Luszetig category

$$KL(G) \simeq (\hat{\mathfrak{g}}, G(\mathcal{O}))\text{-mod}$$

think of as $G[[t]]$

$\hat{\mathfrak{g}}$ is affine Kac-Moody algebra

= central extension of $\mathfrak{g}((t))$

$$\mathfrak{g}((t)) \supset \mathfrak{g}[[t]]$$

so these are $(\mathfrak{g}, K)$ Harish-Chandra modules

but $\mathfrak{g} \to \hat{\mathfrak{g}}$, $K \to G[[t]]$ $\quad$ $\mathrm{Lie}\, K \subseteq \mathfrak{g}$

What is $(\mathfrak{g}, K)$-mod in terms of DAG?

Defn $(\mathfrak{g}, K)$-mod

$= \mathrm{IndCoh}(\hat{G_K})$

$\mathbb{B}K \leftrightarrow K\text{-mod}$

$\mathbb{B}\hat{G_{\{1\}}} \leftrightarrow \mathfrak{g}\text{-mod}$

Let $\mathcal{X} \to \mathcal{Y}$ map of prestacks

$$\mathcal{Y}^{\wedge}_{\mathcal{X}} = \mathcal{X}_{dR} \times_{\mathcal{Y}_{dR}} \mathcal{Y}$$

Exer[1] $\mathcal{X} \hookrightarrow \mathcal{Y}$ is a closed embedding, show that this recovers classical notion

Exer[2] Show $(\mathfrak{g}, K)$-mod $\simeq \mathfrak{g}\text{-mod}^K$ using our definition

FLE: $\mathrm{Whit}(G)_c \simeq KL(\check{G})_{-\check{c}}$

when $c = 0$, $\check{c} = \infty$

$KL(\check{G})_\infty \simeq \mathrm{Rep}(\check{G})$

obj are labeled by dominant wgts of $\check{G}$

= dominant coweights of G

<u>Now</u> What is $\mathrm{Whit}(G)$?

$\mathrm{Whit}(G) := D(\mathrm{Gr}_G)^{N(K),\chi}$ $\qquad K := \mathbb{C}((t))$

what is χ?

$$\begin{array}{ccc} N(K) & \xrightarrow{\chi} & \mathbb{G}_a \\ \downarrow{\scriptstyle ab} & & \uparrow{\scriptstyle \Sigma} \\ (N/[N,N])(K) & \xrightarrow{res} & \prod \mathbb{G}_a \end{array}$$

$G = GL_2$

$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$

$N(K) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C}((t)) \right\}$

$\exists$ exponential D-mod on $\mathbb{A}^1 = \mathbb{G}_a$

(aka Artin-Schrier sheaf) in char p setting

$\chi = \chi^!(\exp)$

$H \xrightarrow{\chi} \mathbb{G}_a$

exp s.t. $\mathrm{add}^!(\exp) = \exp \otimes \exp$ } multiplicative D-module

$\exp = D/D(\partial - 1)$ $\qquad \mathrm{add}: \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$

$H \curvearrowright Y$ $\quad D(Y)^H \underset{Av_*}{\overset{oblv}{\rightleftarrows}} D(Y)$ $\quad$ for H, Y finite type

$\mathcal{F} \in D(Y)$

with $act^* \mathcal{F} \cong \pi^* \mathcal{F}$

$\vdots$

$D(Y)^{H,\chi} \underset{Av_*}{\overset{oblv}{\rightleftarrows}} D(Y)$

$\mathcal{F} \in D(Y)$

w/ $act^* \mathcal{F} \cong \pi^* \mathcal{F} \otimes \chi$

associativity $\leftarrow$ need multiplicative nature of χ

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what is $D(Gr_G)^{N(K)}$?

(1) $N(K) = \bigcup_\alpha N_\alpha$ $\quad G = GL_2$

$N = \mathbb{G}_a$

$N(K) = \mathbb{C}((t))$

$N_\alpha = t^{-\alpha} \mathbb{C}[[t]]$

$\alpha \in \mathbb{Z}_{\geq 0}$

$D(Gr_G) \supset D(Gr_G)^{N(K)} = \bigcap_\alpha D(Gr_G)^{N_\alpha}$

(2) $D(Gr_G)^{N_\alpha}$?

$Gr_G = \bigcup Y_\beta$ $\quad \begin{cases} Y_\beta \text{ f.d.} \\ \text{inv. under } N_\alpha \end{cases}$

$\Rightarrow D(Gr_G)^{N_\alpha} = \varinjlim_\beta D(Y_\beta)^{N_\alpha}$

(3) N_α is still ∞- dim!

$$N_\alpha = \lim_{\gamma} N_{\alpha,\gamma}$$

Ex $G = GL_2$
$N_{\alpha,\gamma}$ truncation of Taylor series.

Then $D(Y_\beta)^{N_\alpha} = D(Y_\beta)^{N_{\alpha,\gamma}}$

$\gamma \gg 0$

$\mathcal{K} = \mathbb{C}((t))$ $\quad \mathcal{O} = \mathbb{C}[[t]]$

Rmk/ consider

$$D(N(\mathcal{K})/N(\mathcal{O}))^{N(\mathcal{K})} \simeq \mathrm{Vect}$$

$$\omega_{Gr_N} \longleftrightarrow 1$$

$G = GL_2$, $N = \mathbb{G}_a = \mathbb{A}^1$
$N(\mathcal{K})/N(\mathcal{O}) = \mathbb{A}^\infty = \bigcup \mathbb{A}^n$
as $\mathbb{A}^n$ is smooth
$\{\omega_{\mathbb{A}^n}\}_{n \in \mathbb{Z}_{\geq 0}}$ $\quad \omega_{\mathbb{A}^n} \simeq \mathcal{O}_{\mathbb{A}^n}[n]$

ω_{Gr_n} is a phantom object
cohomologically trivial,
but non-zero!

$D(Gr_G)^{N(\mathcal{K})}$

Q: $N(\mathcal{K})$-orbits in Gr_G?
a coweight of G, $\lambda \in \Lambda_G$

$\mathbb{G}_m \to T \subset G$ $\quad t \mapsto \lambda(t)$ $\quad t^\lambda \in Gr(G)$

one considers $G_m(\mathcal{K}) \to T(\mathcal{K}) \subset G(\mathcal{K})$

Claim $N(\mathcal{K}) \cdot t^\lambda =: S^\lambda$ $\quad Gr^\lambda = G(O)\, t^\lambda = G(O)\, \lambda(t)\, G(O)$

$\Rightarrow Gr_G = \bigcup_{\lambda \in X_G} S^\lambda$ by Iwasawa decomposition

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$G = GL_2$

$\lambda \in \Lambda_G^+$ (m,n) w/ $m \geq n$

$\lambda \in \check{\Lambda}_G$ (m,n)

Rmk (Geometric Satake)

$H^\cdot(Gr_G^\lambda) \simeq V^\lambda$

$\lambda \in \check{\Lambda}_G^+ \leftrightarrow \lambda \in \Lambda_{\check{G}}^+$

$H^\cdot(S^\mu \cap Gr^\lambda) \longleftrightarrow V_\mu^\lambda \subset V^\lambda$ (pretty amazing)

$D(S^\lambda)^{N(\mathcal{K})} = \mathrm{Vect}$ $\quad \forall \lambda \in \check{\Lambda}_G$

Q: What about $D(S^\lambda_\bullet)^{N(\mathcal{K}), \chi}$?

Claim $D(S^\lambda)^{N(\mathcal{K}), \chi} = \begin{cases} \mathrm{Vect} & \text{if } \lambda \in \check{\Lambda}_G^+ \\ 0 & \text{otherwise} \end{cases}$

$i^\lambda : S^\lambda \hookrightarrow Gr_G$

$D(S^\lambda)^{N(\mathcal{K}), \chi} \underset{(i^\lambda)^!}{\overset{(i^\lambda)_!}{\rightleftarrows}} D(Gr_G)^{N(\mathcal{K}), \chi} = \mathrm{Whit}(G)$

$\mathrm{Whit}(G) \simeq KL(\check{G})_{op} = \mathrm{Rep}(\check{G})$

has objects labeled by

$\check{\Lambda}_G^+ \simeq \Lambda_{\check{G}}^+$

If $H \circlearrowright Y$ transitive, $Y = H/H_1$, $H_1 = \mathrm{Stab}_y(H)$, $y \in Y$

$$D(Y)^{H,\chi} \leftarrow \text{contractible}$$
$$= D(pt)^{H_1, \chi|_{H_1}}$$
$$= \begin{cases} \chi|_{H_1} \text{ is trivial} \Rightarrow \mathrm{Vect} \\ \chi|_{H_1} \text{ is nontrivial} \Rightarrow 0 \end{cases}$$

V w/ $\pi^* V \cong act^* V \otimes \chi|_{H_1}$

~~w/~~ $\pi = act \cdot H_1 \times pt \to pt$

For us:

$H = N(K)$ $\quad Y = Gr_G$ $\qquad y = t^\lambda \quad \lambda \in \check{\Lambda}_G$

$H_1 = \mathrm{Stab}_{t^\lambda}(N(K))$
$= \mathrm{Ad}_{t^\lambda} N(\mathcal{O})$

$\chi|_{H_1}$?

$G = GL_2$ $\qquad \lambda = (m,n)$ $\qquad$ not necessarily dom

$$\begin{pmatrix} t^m & 0 \\ 0 & t^n \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-m} & 0 \\ 0 & t^{-n} \end{pmatrix} = \begin{pmatrix} 1 & b^{n-n} \\ 0 & 1 \end{pmatrix} \xrightarrow{} \chi \qquad m \geq n$$

$$\Rightarrow \chi = 0 \Leftrightarrow m \geq n$$
$$\Leftrightarrow \lambda \text{ is dominant}$$

$$\Rightarrow D(S^\lambda)^{N(K),\chi} = \begin{cases} \mathrm{Vect} & \text{if } \lambda \in \check{\Lambda}^+_G \\ 0 & \text{otherwise} \end{cases}$$